

# Bounds on Seshadri constants on surfaces with Picard number 1.

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## Abstract

In this note we improve a result of Steffens [Ste] on the lower bound for Seshadri constants in very general points of a surface with 1-dimensional Néron-Severi space. We also show a multi-point counterpart of such a lower bound.

## 1 Introduction

Seshadri constants are interesting invariants of big and nef line bundles on algebraic varieties. They capture the so-called local positivity of a given line bundle. Seshadri constants were introduced by Demailly in [Dem]. As a nice introduction to this circle of ideas serves [PAG], an overview of recent result is given in [PSC]. Here we merely recall the basic definition.

**Definition 1.1** *Let  $X$  be a smooth projective variety,  $L$  a big and nef line bundle on  $X$  and  $x \in X$  a point on  $X$ . The number*

$$\varepsilon(L; x) := \inf_{C \ni x} \frac{L \cdot C}{\text{mult}_x C}$$

*is the Seshadri constant of  $L$  at  $x$ .*

By  $\varepsilon(L; 1)$  we denote the maximum

$$\varepsilon(L; 1) := \max_{x \in X} \varepsilon(L; x) \tag{1}$$

of Seshadri constants of  $L$  over all points  $x \in X$ . It is well known (see [PSC, Statement 2.2.8]) that the maximum is attained for very general points  $x \in X$ , i.e. away of a countable union of proper Zariski closed subsets of  $X$ . It is also well known (see [PSC, Proposition 2.1.1]) that there is an upper bound

$$\varepsilon(L; 1) \leq \sqrt[n]{L^n}, \tag{2}$$

where  $n$  is the dimension of  $X$ .

As for lower bounds, Steffens in [Ste, Proposition 1] gave an interesting estimate on  $\varepsilon(L; 1)$  in case that  $X$  is a surface with Picard number 1.

**Proposition 1.2 (Steffens)** *Let  $X$  be a smooth projective surface with Picard number 1 and let  $L$  be the ample generator of the Néron-Severi group of  $X$ . Then*

$$\varepsilon(L; 1) \geq \left\lfloor \sqrt{L^2} \right\rfloor. \tag{3}$$

It is clear that if  $L^2$  is a square, then there is actually an equality

$$\varepsilon(L; 1) = \sqrt{L^2} \quad \text{if } \sqrt{L^2} \in \mathbb{Z}.$$

Our first observation is that only under these circumstances (i.e.  $\sqrt{L^2} \in \mathbb{Z}$ ) is the bound (3) sharp.

For the rest of the paper we write  $N := L^2$  and  $s := \lfloor \sqrt{L^2} \rfloor$ .

Let as before  $X$  be a smooth projective surface with Picard number  $\rho(X) = 1$  and let  $L$  be the ample generator of the Néron-Severi group.

**Lemma 1.3** *If  $L^2$  is not a square, then it is always*

$$\varepsilon(L; 1) > s.$$

*Proof.* We have by assumption that  $s < \sqrt{N}$ , so that

$$s^2 + 1 \leq L^2. \tag{4}$$

Assume to the contrary that  $\varepsilon(L; 1) = s$ . Then for a general point  $x \in X$  there exists a curve  $C \in |pL|$ , for some integer  $p$ , such that

$$pL^2 = sm, \tag{5}$$

where  $m$  denotes as usually the multiplicity of  $C$  at  $x$ .

The curve  $C$  cannot be smooth at  $x$  because then  $pL^2 = s$  could never be satisfied (by our assumption  $L^2 > 1$ ). Hence  $m \geq 2$  and we have by [KSS, Theorem A]

$$m(m - 1) + 1 \leq C^2. \tag{6}$$

Combining (5) and (4) we get

$$ps^2 < ps^2 + p \leq pL^2 = sm,$$

which after dividing by  $s$  and using the fact that  $s$  and  $p$  are integers yields

$$ps \leq m - 1. \tag{7}$$

Now, combining (7) with (6) we get

$$m(m - 1) + 1 \leq C^2 = p^2L^2 = psm \leq (m - 1)m$$

a contradiction.  $\square$

With this fact established, it is natural to ask if there is a lower bound better than  $\lfloor \sqrt{L^2} \rfloor$  if  $L^2$  is not a square. It is not obvious that such a bound exists because there could be a sequence of polarized surfaces  $(X_n, L_n)$  with Picard number 1, such that  $L_n^2 = N$  for all  $n$  and  $\lim_{n \rightarrow \infty} \varepsilon(L_n; 1) = \lfloor \sqrt{N} \rfloor$ . We show that this cannot happen and that there exists a lower bound on  $\varepsilon(L; 1)$  improving that of Steffens in case  $L^2$  is not a square.

## 2 A new lower bound

We introduce some more notation. We assume that  $\sqrt{N}$  is irrational and denote its fractional part by  $\beta$ , thus  $\beta := \sqrt{N} - s > 0$ . We define  $p_0$  as the least integer  $k$  such that  $k \cdot \beta > \frac{1}{2}$ , i.e.

$$p_0 := \left\lceil \frac{1}{2\beta} \right\rceil. \quad (8)$$

Further we set the number  $m_0$  to be equal

$$m_0 := \left\lceil p_0 \cdot \sqrt{N} \right\rceil = p_0 s + \lceil p_0 \beta \rceil = p_0 s + 1. \quad (9)$$

The following theorem is the main result of this note.

**Theorem 2.1** *Let  $X$  be a smooth projective surface with Picard number 1 and let  $L$  be the ample generator of the Néron-Severi space such that  $N = L^2$  is not a square. Then*

$$\varepsilon(L; 1) \geq \frac{p_0}{m_0} N.$$

*Proof.* Note that  $s < \frac{p_0}{m_0} N < \sqrt{N}$ . Indeed, as  $N = (s + \beta)^2$ , we have

$$\frac{p_0}{m_0} N > \frac{p_0}{p_0 s + 1} \cdot (s + 2\beta) \cdot s \geq s.$$

On the other hand

$$\frac{p_0}{m_0} N = \frac{p_0 \sqrt{N}}{\lceil p_0 \sqrt{N} \rceil} \cdot \sqrt{N} < \sqrt{N}.$$

Now, we assume to the contrary that  $\varepsilon(L; 1) < \frac{p_0}{m_0} N$ . Then there exists an integer  $m$  such that for every point  $x \in X$ , there exists a curve  $C_x$  vanishing at  $x$  to order  $\geq m$  (i.e.  $\text{mult}_x C_x \geq m$ ) and

$$\varepsilon(L; x) \leq \frac{L \cdot C}{m} < \frac{p_0}{m_0} N. \quad (10)$$

Such curves  $\{C_x\}$  can be chosen to form an algebraic family and for its arbitrary member  $C$  we have

$$m(m-1) + 1 \leq C^2 \quad (11)$$

by Theorem A in [KSS].

On the other hand there must exist an integer  $p$  such that  $C \in |pL|$ . The condition (10) then translates into

$$\frac{p}{m} < \frac{p_0}{m_0},$$

whereas the inequality (11) requires

$$m(m-1) + 1 \leq p^2 \cdot N. \quad (12)$$

This contradicts Lemma 2.2, which we prove below.  $\square$

We have the following numerical lemma.

**Lemma 2.2** Let  $N$  be a positive integer which is not a square. Let

$$\Omega := \{(p, m) \in \mathbb{Z}_{>0}^2 : m(m-1)+1 \leq Np^2\}$$

and let  $\varepsilon_0 := \min_{(p, m) \in \Omega} \frac{p}{m}$ . Then

$$\varepsilon_0 = \frac{p_0}{m_0}$$

with  $p_0$  and  $m_0$  defined for  $N$  as in (8) and (9).

*Proof.* For the fixed  $p$ , the quotient in question is minimized by the maximal integer  $m$  satisfying the inequality (12). This is

$$m_p := \left\lfloor \frac{1}{2} + \sqrt{Np^2 - \frac{3}{4}} \right\rfloor.$$

We need to show that

$$\frac{p_0}{m_0} \leq \frac{p}{m_p} \quad \text{for all } p.$$

We have certainly

$$\left\lfloor \frac{1}{2} + p\sqrt{N} \right\rfloor \geq \left\lfloor \frac{1}{2} + \sqrt{Np^2 - \frac{3}{4}} \right\rfloor = m_p,$$

so that it is enough to show

$$\frac{p_0}{m_0} \leq \frac{p}{\left\lfloor \frac{1}{2} + p\sqrt{N} \right\rfloor} \quad \text{for all } p. \quad (13)$$

Since

$$p_0 ps + p_0 \left\lfloor \frac{1}{2} + p\beta \right\rfloor = p_0 \left\lfloor \frac{1}{2} + p\sqrt{N} \right\rfloor \quad \text{and} \quad p_0 m_0 = p_0 ps + p$$

inequality (13) would follow from

$$p \geq p_0 \left\lfloor \frac{1}{2} + p\beta \right\rfloor. \quad (14)$$

For  $p < p_0$  the right hand side of (14) is zero, since then  $p\beta < \frac{1}{2}$ . For  $p \geq p_0$  we write  $p = qp_0 + r$  with  $q \geq 1$  and  $0 \leq r \leq p_0 - 1$ . In particular  $r\beta < \frac{1}{2}$ , so that

$$p_0 \left\lfloor \frac{1}{2} + p\beta \right\rfloor = p_0 \left\lfloor \frac{1}{2} + r\beta + qp_0\beta \right\rfloor \leq p_0 \lfloor qp_0\beta + 1 \rfloor \leq p_0 q \leq p.$$

The last but one inequality holds because  $p_0\beta < 1$  and  $q \geq 1$ . This verifies (14) and the proof is finished.  $\square$

The next example shows that in some situations our bound is optimal.

**Example 2.3** Let  $N = 2d$  be an even integer such that  $N+1 = \ell^2$  is a square. A general abelian surface  $X$  with polarization  $L$  of type  $(1, d)$  has Picard number equal 1, see e.g. [CAV, Section 9.9]. For such surfaces we know by [Bau, Theorem 6.1] that

$$\varepsilon(L; 1) = \frac{2d}{\ell} = \frac{1}{\sqrt{N+1}} N.$$

On the other hand we have  $p_0 = 1$  and  $m_0 = \left\lfloor \sqrt{N} \right\rfloor + 1 = \ell$ , so that

$$\varepsilon(L; 1) = \frac{p_0}{m_0} N$$

in that case.

In general we expect however that  $\varepsilon(L; 1)$  on surfaces with Picard number 1 is subject to a much stronger numerical restriction.

**Conjecture 2.4** *Let  $X$  be a smooth projective surface with Picard number 1 and let  $L$  be the ample generator of the Néron-Severi space with  $N = L^2$ . Then*

$$\varepsilon(L; 1) \geq \begin{cases} \sqrt{N} & \text{if } N \text{ is a square} \\ \frac{Nk_0}{\ell_0} & \text{if } N \text{ is not a square} \end{cases}$$

and  $(\ell_0, k_0)$  is the primitive solution of Pell's equation

$$\ell^2 - Nk^2 = 1.$$

The inequality in Theorem 2.1 can be viewed as the next step (after Steffens) towards approximating  $\sqrt{N}$  by continued fractions.

### 3 Multi-point Seshadri constants

In the last paragraph we show that a lower bound of Steffens type can be given also for multi-point Seshadri constants. This is a variant of Definition 1.1 due to Xu, see [X94].

**Definition 3.1** *Let  $X$  be a smooth projective variety,  $L$  a big and nef line bundle on  $X$ ,  $r \geq 1$  an integer and  $x_1, \dots, x_r$  distinct points on  $X$ . The real number*

$$\varepsilon(L; x_1, \dots, x_r) := \inf_{C \cap \{x_1, \dots, x_r\} \neq \emptyset} \frac{L \cdot C}{\text{mult}_{x_1} C + \dots + \text{mult}_{x_r} C}$$

is the multi-point Seshadri constant of  $L$  at points  $x_1, \dots, x_r$ .

The interest in these numbers comes from the fact that, at least conjecturally, their behavior is more predictable than that of their one-point cousins. We refer again to [PSC, Sections 2 and 6] for introduction to that circle of ideas.

Similarly to (1) we set

$$\varepsilon(L; r) := \max_{\{x_1, \dots, x_r\} \subset X} \varepsilon(L; x_1, \dots, x_r).$$

The following result parallels Proposition 1.2.

**Theorem 3.2** *Let  $X$  be a smooth projective surface with Picard number 1 and let  $L$  be the ample generator of the Néron-Severi group of  $X$ . Then*

$$\varepsilon(L; r) \geq \left\lfloor \sqrt{\frac{L^2}{r}} \right\rfloor.$$

*Proof.* We denote  $s := \left\lfloor \sqrt{\frac{L^2}{r}} \right\rfloor$  and assume to the contrary that  $\varepsilon(L; r) < s$ . Then for arbitrary  $x_1, \dots, x_r$  there are irreducible curves  $C_{x_1, \dots, x_r}$  such that

$$\frac{L \cdot C_{x_1, \dots, x_r}}{\sum_{i=1}^r \text{mult}_{x_i} C_{x_1, \dots, x_r}} < s. \quad (15)$$

One can choose these curves to move in an algebraic family. As the Picard number of  $X$  is 1, there is in fact an integer  $p$  such that this family is a subset of the linear series  $pL$ . If  $m_1, \dots, m_r$  are positive integers such that

$$\text{mult}_{x_i} C_{x_1, \dots, x_r} \geq m_i \quad \text{for all } i = 1, \dots, r,$$

then for any member  $C$  of the family we have by [X94, Lemma 1]

$$C^2 \geq m_1^2 + \dots + m_{r-1}^2 + m_r(m_r - 1). \quad (16)$$

We can renumber the points so that  $m_r \leq m_i$  for all  $i = 1, \dots, r$ . The inequality (15) implies that

$$rps^2 \leq pL^2 < s \cdot \sum_{i=1}^r m_i.$$

Dividing by  $s$  and taking into account that all involved numbers are integers we obtain

$$rps \leq \sum_{i=1}^r m_i - 1. \quad (17)$$

On the other hand from (16), (15) and (17) we have

$$\sum_{i=1}^r m_i^2 - m_r \leq pL \cdot C < ps \sum_{i=1}^r m_i \leq \frac{1}{r} \sum_{i=1}^r m_i \left( \sum_{i=1}^r m_i - 1 \right) \leq \sum_{i=1}^r m_i^2 - m_r$$

which gives the desired contradiction.  $\square$

We have the following straightforward corollary.

**Corollary 3.3** *Let  $X$  and  $L$  be as in Theorem 3.2. Assume that the degree of  $L$  is of the form  $L^2 = rd^2$  for some positive integer  $d$ . Then we have the equality*

$$\varepsilon(L; r) = d.$$

**Remark 3.4** The same statement was proved in [PSC, Theorem 6.1.10] for surfaces with arbitrary Picard number under the additional assumption that  $L$  is very ample. It is expected that the equality

$$\varepsilon(L; r) = \sqrt{\frac{L^2}{r}}$$

holds on arbitrary surfaces, provided  $r$  is sufficiently large. This is a natural generalization of Nagata Conjecture as explained in detail in [Sze].

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